How Newton Was Misled by Kepler: A Classical Derivation of Newtonian Gravitation from Precessing Orbits

Pierre Berrigan

OpenSci.World, Montreal, Canada
ORCID ID: 0009-0001-4010-3384
pierre.berrigan@OpenSci.World

April 29, 2024

Abstract

The orbits of planets around the Sun are observed to be hypotrochoidal instead of the closed elliptical trajectories initially proposed by Johannes Kepler. This deviation was dubbed "relativistic" since Albert Einstein proposed an explanation based on the theory of General Relativity. However, with apsidal precession as an empirical property of all orbiting bodies, an exact solution of the Newtonian law of gravitation is produced which predicts hypotrochoidal orbits. We show that empirical observations, basic algebra, and parameters obtained from natural values match the predictions from General Relativity with great precision.

Keywords— Newton, Kepler, gravitation, precession, relativity, orbits, planets

1 Introduction: how Newton was misled by Kepler

It was the year 1609 when Johannes Kepler enunciated his first law of planetary motion, establishing the trajectories of planets in orbit around the Sun as ellipses with the Sun occupying one of their foci. This empirical law was based on a lifetime of astronomical observations by him and his predecessor and mentor, Tycho Brahé.

It wasn't until 250 years later, in 1859, that further, more precise observations, mainly of Mercury's orbit by Urbain LeVerrier[1], revealed that instead of being a perfectly closed elliptical trajectory, the orbit of Mercury was being shifted forward on each of its iterations, albeit by a very minute although incontrovertible angle.

What was then dubbed the "anomalous precession of the perihelion" of Mercury's orbit was later shown not to be "anomalous" at all, but rather an inherent characteristic of all orbiting bodies. Such a precessing orbit is commonly represented as an elliptical figure that rotates about its own gravitating focus.[1]

Representations are useful. However, in physical reality, orbital trajectories are not solid objects that can be given motion. Trajectories "occur", they do not "exist" as physical objects, which is a prior requirement for things to be endowed of motion. Instead, the orbiting bodies themselves should be considered to follow hypotrochoidal instead of elliptical trajectories, with a positive precession.[2]

Given such glaring evidence, a question that one is justified to ponder is: how would Kepler formulate his first law of planetary motion, had sufficiently precise data been available to him? Furthermore, how would Isaac Newton, some 80 years after Kepler, derive his universal law of gravitation if Kepler's first law stated that orbits had the shape of hypotrochoids?

The fact of the matter is that Newton, indeed, had studied what he had dubbed "revolving orbits", in Principia's first book, Proposition 44 and corollaries[2]. He determined that for an elliptical orbit to be itself in rotation about its gravitation focus, the orbiting body should be subject to an acceleration towards that focus by means of a force inversely proportional to the cube of the distance, in addition to the already-determined inverse square law of gravitational forces[3].

[1] Example of how apsidal precession is commonly represented: https://tinyurl.com/5fxkpwjs

[2] How apsidal precession should be represented: https://tinyurl.com/bddwvus3

[3] "The difference of the forces, by which two bodies may be made, to move equally, one in a quiescent, the other in the same orbit revolving, is in a triplicate ratio of their common altitudes inversely."

Newton's underlying motivation for Proposition 44, however, was an attempt to model orbital trajectories that deviated from the perfect close ellipse by the interfering action of other planets, a well-known phenomenon that, incidentally, also results in a forward precession of the periapsis. As such, it was never considered as anything more than an interesting curiosity by scholars of the following centuries. Before LeVerrier's measurements, there was no reason to suspect orbits to inherently be anything other than ellipses, and Newton's universal law of gravitation to be anything other than an inverse square function of distance.

2 Hypotrochoids

Elliptical conic sections (figure 1) can be defined using only two parameters: a semi-major axis, commonly referred to as a, and a semi-minor axis, usually b.

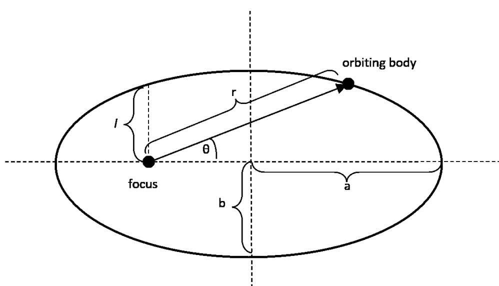

Figure 1: *Schematic representation of an ellipse.*

From those two parameters, additional characteristics of ellipses are defined, such as:

- the eccentricity as: $\epsilon = \sqrt{1 - \frac{b^2}{a^2}}$
- the semi-latus rectum as: $l = \frac{b^2}{a}$

To make the study of orbital trajectories practical, ellipses can be represented mathematically in polar form as:

$$r = \frac{l}{1 - \epsilon cos(\theta)}$$

where r is the distance from the orbited focus to the orbiting body at an angle of θ. For the figure to instead be a hypotrochoid, one parameter must be added: ω which indicates the extent to which the ellipse precesses at every iteration of the orbital cycle. Thus:

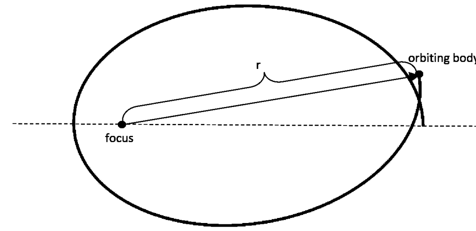

Figure 2: *Hypotrochoidal trajectory of orbiting body after $\theta = \frac{2\pi}{\omega}$ with $0 < \omega < 1$*

$$r = \frac{l}{1 - \epsilon cos(\omega\theta)} \tag{1}$$

is a mathematical representation of a hypotrochoid (figure 2) in polar notation. A value of ω greater than 1 will make the ellipse precess backward, whereas $0 < \omega < 1$ will produce a forward precession.

It is useful to note that to complete one full orbital revolution, the line that links the focus to the orbiting body must vary by an angle of $\frac{2\pi}{\omega}$.

3 Kepler's laws.

3.1 Kepler's first law.

In accordance with the arguments given in paragraph 1, empirical observations require Kepler's first law to be rewritten as follows:

> *"All planets move about the Sun in hypotrochoidal orbits with forward precession, having the Sun as one of the foci."*

3.2 Kepler's second law

Kepler's second law of planetary motions states that an imaginary line linking the orbiting body to the focus sweeps equal areas during equal intervals of time.

Orbital periods are empirically measured as the time it takes for a planet to travel from one particular position to the same position in the following orbit. Despite the extra distance travelled on a hypotrochoidal trajectory compared to an ellipse, the period is not observed to be longer. Thus, the usual formulation for Kepler's second law holds:

> *"A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time."*

An equivalent definition is: the ratio of the area swept by the said line over the time it takes for this area to be swept is a constant.

This is true regardless of the time interval considered, which can be anything between infinitesimal and one entire revolution. If, for the gravitating body to travel one entire orbit, the area swept is denoted by A and the time it takes is denoted by T, then:

$$\frac{dA}{dT} = \kappa \tag{2}$$

is a constant.

We know[3] that the area of a circular sector of radius r sustained by a central angle θ is $\frac{1}{2}r^2\theta$. Hence:

$$dA = \frac{1}{2}r^2 d\theta$$

Therefore, equation 2 can be written:

$$\frac{dA}{dT} = \frac{1}{2}r^2\frac{d\theta}{dT} = \kappa \tag{3}$$

3.3 Kepler's third law

> *"The square of a planet's orbital period is proportional to the cube of the semi-major axis of its orbit."*

Although orbits have been determined by observation to be hypotrochoids instead of ellipses, no empirical data justify any change to Kepler's third law. Thus, the usual formulation for the period T remains:

$$T = 2\pi\sqrt{\frac{a^3}{GM}} \tag{4}$$

where M is the mass of the object about which planets orbit, and G is the universal gravitational constant, which allows to make a mathematical equality from a statement of proportionality.

This enables us to represent the ratio of area swept to period for one entire orbit. Hypotrochoids are open geometrical figures, and their area is impossible to define. However, Kepler's second law relates to "area swept", which happens to equate to "area" only for close figures such as ellipses. For hypotrochoids, the "area swept" can still be calculated.

As stated earlier, one entire orbital revolution is completed when the angle of the line segment has travelled $\frac{2\pi}{\omega}$ radians. Therefore, area swept is:

$$A = \frac{1}{2}\int_0^{\frac{2\pi}{\omega}} r^2 d\theta = \frac{\pi l^2}{\omega\sqrt{(1-\epsilon^2)^3}} \tag{5}$$

From equations (4) and (5), we get:

$$\frac{A}{T} = \frac{l^2}{2\omega\sqrt{\frac{a^3}{GM}}\sqrt{(1-\epsilon^2)^3}} = \kappa \tag{6}$$

which is the same constant as in equation (2).

3.4 Modified Keplerian Dynamics (MOKD)

We shall hereafter refer to Kepler's laws as enounced above as: *Modified Keplerian Dynamics*, abbreviated as: *MOKD*.

4 Newton's universal law of gravitation.

To derive his universal law of gravitation, also known as the "inverse square law", Newton used all three of Kepler's laws of planetary motion. In this section, we will again follow Newton's reasoning but instead use the hypotrochoidal trajectories of orbits as Kepler's first law.

4.1 Exact solution of gravitation with MOKD

The acceleration of an orbiting test mass will be found by differentiating the equation of motion twice over time. From equation (1):

$$\frac{dr}{dT} = -\frac{\epsilon l \omega sin(\omega\theta)}{(1-\epsilon cos(\omega\theta))^2}\frac{d\theta}{dT} \tag{7}$$

From equations (3) and (1):

$$\frac{d\theta}{dT} = \frac{2\kappa}{r^2} = \frac{2\kappa(1-\epsilon cos(\omega\theta))^2}{l^2} \tag{8}$$

Replacing $\frac{d\theta}{dT}$ of equation (7) by equation (8) :

$$\frac{dr}{dT} = -\frac{2\kappa\omega\epsilon sin(\omega\theta)}{l} \tag{9}$$

Differentiating a second time:

$$\frac{d^2r}{dT} = -\frac{2\kappa\omega^2\epsilon cos(\omega\theta)}{l}\frac{d\theta}{dT} \tag{10}$$

Again using equation (8) to solve $\frac{d\theta}{dT}$:

$$\frac{d^2r}{dT} = -\frac{4\kappa^2\omega^2\epsilon cos(\omega\theta)(1-\epsilon cos(\omega\theta))^2}{l^3} \tag{11}$$

and using equation (1) to eliminate θ:

$$\frac{d^2r}{dT} = 4\kappa^2\omega^2(\frac{1}{lr^2} - \frac{1}{r^3}) = a_{tot} \tag{12}$$

Already, we see appearing the additional r^{-3} term calculated by Newton in Proposition 44; however, in the present case, no interfering influence from other planets is involved.

Equation (12) gives the total acceleration a_{tot} the test mass is subject to in its orbital trajectory, which amounts to the sum of the radial and the centripetal components of acceleration. However, only the radial component of acceleration is of interest. Therefore, we subtract from a_{tot} the centripetal component of acceleration a_c, which is given by:

$$a_c = r\left(\frac{d\theta}{dT}\right)^2 \tag{13}$$

and use Newton's second law of motion to calculate the radial force[4]:

$$\begin{aligned} F = m(a_{tot} - a_c) &= m\left(\frac{d^2r}{dT} - r\left(\frac{d\theta}{dT}\right)^2\right) \\ &= 4\kappa^2 m\left(\frac{\omega^2}{lr^2} + \frac{1-\omega^2}{r^3}\right) \end{aligned} \tag{14}$$

From equation (6):

$$\begin{aligned} \kappa^2 = \left(\frac{A}{T}\right)^2 &= \frac{GMl^4}{4a^3\omega^2\left(1-\epsilon^2\right)^3} = \frac{GMl^4}{4a^3\omega^2\left(\frac{l}{a}\right)^3} \\ &= \frac{GMl}{4\omega^2} \end{aligned} \tag{15}$$

equation (14) becomes:

$$F = GMm\left(\frac{1}{r^2} + \frac{l\left(1-\omega^2\right)}{\omega^2 r^3}\right) \tag{16}$$

Let:

$$\Omega = \frac{l\left(1-\omega^2\right)}{\omega^2} \tag{17}$$

The universal law of gravitation becomes formulated as:

$$F = GMm\left(\frac{1}{r^2} + \frac{\Omega}{r^3}\right) \tag{18}$$

4.2 Finding the value of Ω

In Table 1, we find the values of ω as measured[5, 6, 7] by observations of the orbits of the solar system's planets. When represented in units of Ω as per equation (17), we obtain a value that is constant to within 1, 6% of average, when excluding the data for Saturn, which is known[8] to exhibit a precession that is unexplained by Newtonian and even General Relativistic predictions.

However, it is evident that such a way of establishing the value of Ω would yield a result valid solely for the solar system. For the Newtonian law of gravitation to truly be universal while remaining empirical, some observations of precessions for planets in orbit around stars other than the Sun must intervene. Unfortunately, at the time of writing the present article, the only such observation in existence is that of star S2[9] which orbits the supermassive body at the midst of the Milky Way.

Orbiting body	Central mass (Kg)	l (m)	ω	Ω	G_1 (m·kg^{-1})
Mercury	1.99E+30	5.54E+10	0.9(7)19905161	8882.18	4.47E-27
Venus	1.99E+30	1.08E+11	0.9(7)59060114	8855.77	4.45E-27
Earth	1.99E+30	1.50E+11	0.9(7)70380402	8859.71	4.45E-27
Mars	1.99E+30	2.26E+11	0.9(7)80322377	8913.40	4.48E-27
Jupiter	1.99E+30	7.76E+11	0.9(7)94490741	8555.61	4.30E-27
Saturn	1.99E+30	1.42E+12	0.9(7)97613762	6789.14	3.41E-27
S2	8.48E+36	3.15E+13	0.9(3)52677214	35061740199.03	4.14E-27
				Avg:	4.38E-27

Table 1: *Measurements of Ω from observations of the apsidal precession for the planets of the solar system and star S2.*

Although, admittedly, a data set of only two individuals is rather scarce in terms of statistical population, the numbers nevertheless show a proportionality of Ω with the mass of the central body about which the orbiting bodies revolve. Thus, by replacing Ω with its ratio to the mass of the central body, we obtain a value that is constant to within 3% of the average of $4,38E^{-27}$. Noticing that $\frac{\Omega}{M}$ is somewhat constant, equation (18) can be rewritten as:

$$F = GMm\left(\frac{1}{r^2} + \frac{G_1 M}{r^3}\right) \tag{19}$$

where $G_1 = \frac{\Omega}{M}$ is the constant for which we seek the value. Incidentally, the value of G_1 coincides within $1,6\%$ to a natural value of:

$$G_1 \approx \frac{6G}{c^2} \pm 1,59\% \tag{20}$$

that is: six times the gravitational constant divided by the speed of light squared, which also happens to be in the correct units of metres per kilogram.

4.3 Final forms

Inserting the value of G_1 from equation (20) into equation (19), the equation of force takes the form:

$$F = GMm\left(\frac{1}{r^2} + \frac{6GM}{c^2 r^3}\right) \tag{21}$$

Using equation (20) to solve equation (17) for ω leads to a planet's precession in ratio of 2π as:

$$\omega \approx \sqrt{\frac{l}{l + \frac{6GM}{c^2 l}}} \tag{22}$$

The units of σ which is measured in radian per revolution converts to the units of ω in fraction of 2π by:

$$\omega = \frac{1}{\frac{\sigma}{2\pi} + 1} \tag{23}$$

Therefore, from equations (22) and (23), precession expressed as radians per revolution is:

$$\sigma \approx 2\pi\left(\sqrt{1 + \frac{6GM}{c^2 l}} - 1\right) \tag{24}$$

5 Precession in general relativity

From General Relativity's postulate of the constancy of the speed of light, Einstein came up in 1915[10] with a solution to Mercury's "anomalous" precession, which in its first-order approximation is as follows:

$$\sigma = \frac{24\pi^3 a^2}{T^2 c^2 \left(1 - \epsilon^2\right)} \tag{25}$$

where, in addition to the symbols defined earlier, c is the speed of light, and σ is the precession in units of radians per revolution. Equation (25) reduces to:

$$\sigma = \frac{6\pi\mu}{c^2 l} \tag{26}$$

where μ is the product of the mass of the body about which planets orbit and the universal gravitational constant. As we can see, nothing in this formula is particular to Mercury or its orbit; the calculated precession depends only on the orbit's semi-latus rectum and the mass of the orbited body. Insofar as General Relativity's prediction is accurate, this indicates that in prin-

Comment	M/l	σ (GR)	σ (MOKD)	Difference	Orbital velocity (%c)
Sun-Mercury	3.44E+19	4.81E-07	4.81E-07	0.00%	0.02%
	2.00E+20	2.80E-06	2.80E-06	0.00%	0.04%
	1.00E+21	1.40E-05	1.40E-05	0.00%	0.09%
	5.00E+21	6.99E-05	6.99E-05	0.00%	0.19%
	2.50E+22	3.50E-04	3.50E-04	0.00%	0.43%
White dwarf	1.25E+23	1.75E-03	1.75E-03	0.01%	0.96%
S2	2.69E+23	3.77E-03	3.76E-03	0.03%	1.41%
	6.25E+23	8.74E-03	8.74E-03	0.07%	2.15%
	2.00E+24	2.80E-02	2.79E-02	0.22%	3.86%
	4.00E+24	5.60E-02	5.57E-02	0.44%	5.46%
	6.00E+24	8.39E-02	8.34E-02	0.66%	6.70%
Neutron star	**3.00E+25**	**4.20E-01**	**4.07E-01**	**3.13%**	**15.26%**
	1.25E+26	1.75E+00	1.56E+00	11.02%	33.75%
ISCO black hole	2.25E+26	3.14E+00	2.60E+00	17.16%	50.00%
Unstable orbit	3.37E+26	4.71E+00	3.65E+00	22.51%	70.71%
Photon sphere	4.49E+26	6.28E+00	4.60E+00	26.79%	100.00%

Table 2: *Comparative calculations of precession from General Relativity and Modified Keplerian Dynamics (MOKD)*

ciple, any orbiting body should normally precess, and not just Mercury!

5.1 Comparing MOKD and GR

Table 2 shows the calculated precession for General Relativity (from equation (26)) and for Modified Keplerian Dynamics (from equation (24)) for values of $\frac{M}{l}$ increasing (larger M and/or smaller l) from the Sun-Mercury system to a maximum of impossible orbits below the innermost stable circular orbit (ISCO) around a hypothetical black hole. We see that the precession calculated by MOKD fits that from General Relativity to within 1% until $\frac{M}{l}$ reaches values such that orbital velocities approach a significant percentage of the speed of light.

Such extreme conditions correspond to a hypothetical planet orbiting at a very low altitude around a central mass akin to a neutron star (the value of $\frac{M}{l}$ of $3.00E^{+25}$ in the highlighted line of Table 2 is for a planet at a 100-km radius orbit around a 2 $M_{\odot}$ body). Even for star S2, which is the most extreme case observed to date with a $\frac{M}{l}$ of $2.69E^{+23}$, the difference between the predictions from MOKD and General Relativity is no more than 0.03%. This suggests that Einstein's prediction of precession could not have been confirmed by observation until measurements could be made for a stellar system of similar configuration.

6 Discussion

We argue that as early as the 16th century, Kepler had all the necessary elements in hand to formulate his first law of planetary motion as hypotrochoidal instead of elliptical trajectories, barring the possible exception of sufficient precision in the measurements. On this basis, Newton would have established a universal law of gravitation that would have approached very closely the predictions of orbital mechanics from General Relativity. We claim that the procedure described in the current paper is similar to what Newton could have done.

Attempts to explain Mercury's apsidal precession using Newtonian mechanics, such as those from Hall[11], Seeliger[12], Wells's characters 'Alice' and 'Bob'[13] and others, have been using a top-down approach by hypothesizing various imagined modifications to the mathematics of the universal law of gravitation until one is found that closely predicted the observed apsidal precession of Mercury. However, the "let's-try-this-and-see-how-it-goes" approach stems more from guesswork, and since many different solutions may lead to the same prediction, it is impossible this way to ascertain which solution can potentially be the correct one.

Gerber's[14] and Einstein's approaches were similarly top-down, however, their hypotheses (finite speed of gravity; curved spacetime) were related to the physical world, and not just to the mathematical model of Newtonian gravitation. In contrast, the derivation presented here used a bottom-up approach, based solely on the observable fact that orbits precess, consistent with Newton's "Hypotheses non fingo". Of course, the present derivation pretends in no way to be better or to replace General Relativity's model, nor does it pretend to explain what gravity is.

7 Conclusion

Had he had access to measurements of planetary orbits with a precision in the order of LeVerrier's, Johannes Kepler could have established the trajectory of orbits as elliptical hypotrochoids instead of perfect ellipses as the statement of his first law of planetary orbits. Subsequently, Newton would have enunciated the universal law of gravitation as:

$$F = GMm\left(\frac{1}{r^2} + \frac{6GM}{c^2 r^3}\right)$$

which predicts such orbital trajectories. The derivation shown in this article depicts what Newton could have come up with and how he could have done it.

The implication is that the observation of Mercury's orbit in 1859 by LeVerrier would have been no mystery, but rather a drab confirmation of what should be expected. Hence, the explanation of the "anomalous apsidal precession of Mercury" could not have been considered the grandiosely successful test of general relativity that history tells us it is... In fact, the necessary astronomical technologies might still be decades away before some spiritual descendant of LeVerrier could observe some apsidal precession that shows a discrepancy from Newton's prediction, thus confirming Einstein's solution of precession.

It is nowadays generally accepted that the apsidal precession said "relativistic" is due to "spacetime" being curved in the vicinity of massive bodies. It is peculiar that such a firmly established physical property of the real world would be needed as an explanation for a discrepancy in a physical phenomenon (gravity) that itself has no explanation in the first place. It would seem that Newton's inverse square law is so deeply anchored in the paradigm of physics, that any deviation from it would necessarily require a physical explanation from the realm of "new physics". Had the universal law of gravitation instead been formulated as an "inverse square plus an inverse cube" function of distance in the first place, no doubt that the evolution of physics in the last two centuries could have been very different.

References

[1] Le Verrier U, 1859 "Lettre de M. Le Verrier à M. Faye sur la théorie de Mercure et sur le mouvement du périhélie de cette planète" *C. r. hebd. séances Acad. sci* **49** 379-383

[2] Cohen B, Whitman A and Budenz J. 1999 *The Principia: The Authoritative Translation and Guide: Mathematical Principles of Natural Philosophy* (University of California Press)

[3] Tennenbaum M and Pollard H, 1985 *Ordinary Differential Equations* (Dover Publications) p 494

[4] Tennenbaum M and Pollard H, 1985 *Ordinary Differential Equations* (Dover Publications) p 495

[5] Duncombe R L, 1956 "Relativity effects for the three inner planets." *AJ* **61** 174

[6] Biswas A and Mani K, 2008 "Relativistic perihelion precession of orbits of Venus and the Earth" *Cent. Eur. J. Phys* **6(3)** 754-758

[7] Nyambuya G G, 2015 "Azimuthally symmetric theory of gravitation – II. On the perihelion precession of solar planetary orbits" *MNRAS* **491(3)** 3034–3043

[8] Iorio L, 2009 "The Recetly Determined Anomalous Perihelion Precession of Saturn" *AJ* **137(3)** 3615

[9] Abuter R et al, 2020 "Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole" *A&A* **636** L5

[10] Einstein A, 1915 "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie." *Abh. K. Preuss. Akad. Wiss.* **SPAW** 831–839

[11] Hall A, 1894 "A Suggestion in the Theory of Mercury" *AJ* **319** 49

[12] von Seeliger H, 1915 "Über die Anomalien in der Bewegung der inneren Planeten." *Astron. Nachr.* **201** 273–280

[13] Wells J D, 2016 "When effective theories predict: the inevitability of Mercury's anomalous perihelion precession" arXiv:1106.1568[physics.hist-ph]

[14] Gerber P, 1917 " Die Fortpflanzungsgeschwindigkeit der Gravitation" *AdP* **52** 415